

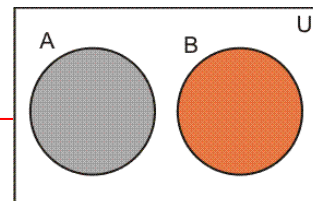
Introduction to Principle of Inclusion and Exclusion (PIE)

排容原理/容斥原理

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The main concept of Principle of Inclusion and Exclusion is simple: *If you add more than needed, you deduct; and if you deduct more than needed, you add.* In Chinese:“多退少補”.

Before we discuss **PIE**, let us begin something easier first:



Preliminary theorems

(1) If A, B are *disjoint* sets, that is $A \cap B = \emptyset$ (\emptyset is called the *empty set*), then
 $|A \cup B| = |A| + |B|$, where $|A|$ denotes the number of elements in A.

(2) Also if U denotes the *universal set*, then

$$|A' \cap B'| = |U| - |A| - |B|,$$

where A' denotes the *complement* of A, that is, A' contains elements not in A, but in the universal set U.

(3) If A, B are events in the sample space U, then the probabilities can be calculated as:

$$P(A \cup B) = P(A) + P(B)$$

$$P(A' \cap B') = 1 - P(A) - P(B)$$

Example 1 :

Let $U = \{ a, b, c, d, e, f, g, h, i, j, k \}$, $A = \{ a, b, c \}$, $B = \{ d, e, f, g \}$

Use the data to verify that the above formulas for

(a) $|A \cup B|$ (b) $|A' \cap B'|$ (c) $P(A \cup B)$ (d) $P(A' \cap B')$

are correct.

Solution

$$A \cup B = \{ a, b, c, d, e, f, g \} \quad A \cap B = \emptyset$$

(a) $|A \cup B| = 7$, $|A| + |B| = 3 + 4 = 7 \quad \therefore |A \cup B| = |A| + |B|$

(b) $A' \cap B' = \{ d, e, f, g, h, i, j, k \} \cap \{ a, b, c, h, i, j, k \} = \{ h, i, j, k \}$, $|A' \cap B'| = 4$

$$|U| - |A| - |B| = 11 - 3 - 4 = 4$$

$$\therefore |A' \cap B'| = |U| - |A| - |B|$$

(c) $P(A \cup B) = \frac{7}{11}$, $P(A) + P(B) = \frac{3}{11} + \frac{4}{11} = \frac{7}{11}$

$$\therefore P(A \cup B) = P(A) + P(B)$$

(d) $P(A' \cap B') = \frac{4}{11}$, $1 - P(A) - P(B) = 1 - \frac{3}{11} - \frac{4}{11} = \frac{4}{11}$

$$\therefore P(A' \cap B') = 1 - P(A) - P(B)$$

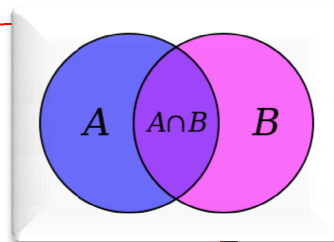
Principle of Inclusion and Exclusion (PIE)

PIE occurs in *dual form* :

$$(1) \quad |A \cup B| = |A| + |B| - |A \cap B|$$
$$|A' \cap B'| = |U| - |A| - |B| + |A \cup B|$$

(2) If A, B are events in the sample space U, then the probabilities can be calculated as:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A' \cap B') = 1 - P(A) - P(B) + P(A \cup B)$$



Example 2 :

- (a) Find the number of integers from 1 through 1000 that are multiples of 3 or multiples of 5.
- (b) If a number is chosen from integers 1 through 1000, find the probability that the number is neither a multiple of 3 nor a multiple of 5.
- (c) Find the **sum** of all integers from 1 through 1000 that are multiples of 3 or multiples of 5.

Solution

- (a) Let $A = \{ \text{integers from 1 through 1000 that are multiples of 3} \}$
 $B = \{ \text{integers from 1 through 1000 that are multiples of 5} \}$
 $A \cap B = \{ \text{integers from 1 through 1000 that are multiples of 15} \}$
 $A \cup B = \{ \text{integers from 1 through 1000 that are multiples of 3 or multiples of 5} \}$
Denote $[x]$ be the greatest integer smaller or equal to x . We have :

$$|A| = \left[\frac{1000}{3} \right] = 333, \quad |B| = \left[\frac{1000}{5} \right] = 200, \quad |A \cap B| = \left[\frac{1000}{15} \right] = 66$$

$$\therefore |A \cup B| = |A| + |B| - |A \cap B| = 333 + 200 - 66 = \underline{\underline{467}}$$

- (b) Using same notation as in (a),

$$A' \cap B' = \{ \text{the number is neither a multiple of 3 nor a multiple of 5} \}$$

$$P(A' \cap B') = 1 - P(A) - P(B) + P(A \cup B) = 1 - \frac{333}{1000} - \frac{200}{1000} + \frac{467}{1000} = \frac{934}{1000} = \underline{\underline{\frac{467}{500}}}$$

- (c) Let $A = \{ \text{sum of all integers from 1 through 1000 that are multiples of 3} \}$
 $B = \{ \text{sum of all integers from 1 through 1000 that are multiples of 5} \}$

$$|A \cup B| = |A| + |B| - |A \cap B| = \frac{333(3+999)}{2} + \frac{200(5+1000)}{2} - \frac{66(15+990)}{2}$$

$$= 166833 + 100500 - 33165 = \underline{\underline{84168}}$$

Example 3 :

Six balls A, B, C, D, E and F are put into six boxes numbered 1, 2, 3, 4, 5, and 6. If only one ball can be put into each box, what is the probability that ball A is not in box 1 and ball B is not in box 6?

Solution

Let $U = \{\text{Any possible combinations of putting balls in boxes}\}$

$X = \{\text{ball A is in box 1}\}$

$Y = \{\text{ball B is in box 2}\}$

$$P(X' \cap Y') = 1 - P(X) - P(Y) + P(X \cup Y)$$

$$= 1 - \frac{5!}{6!} - \frac{5!}{6!} + \frac{4!}{6!} = 1 - \frac{1}{6} - \frac{1}{6} + \frac{1}{30} = \underline{\underline{\frac{7}{10}}}$$

Example 4 :

In a survey of 1000 students in Queen's College, 280 were listed as drinking tea, 275 as drinking coffee and 95 drank both tea and coffee at lunch yesterday. Find how many students

- (a) drank neither tea or coffee.
- (b) drank only tea
- (c) drank only coffee.

Solution:

Let $|T|$ denote the number of students who drank tea and
 $|C|$ denote the number of students who drank coffee.

By **PIE**, we have

$$|T \cup C| = |T| + |C| - |T \cap C|$$

From the given question we have $|T| = 280$, $|C| = 275$ and $|T \cap C| = 95$

Therefore,

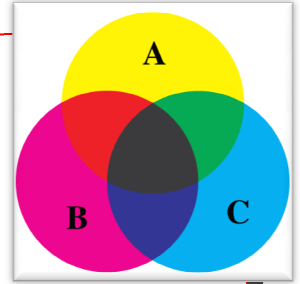
$$\begin{aligned} |T \cup C| &= |T| + |C| - |T \cap C| \\ &= 280 + 275 - 95 = 460 \end{aligned}$$

$$\therefore \text{Number of students who drank tea or coffee} = 460$$

Since the total number of students taking the survey in Queen's College = 1000

- (a) Total number of students who drank neither tea or coffee = $1000 - 460 = \underline{\underline{540}}$
- (b) Number of students who drank only tea = $280 - 95 = \underline{\underline{185}}$
- (c) Number of students who drank only coffee = $275 - 95 = \underline{\underline{180}}$

Principle of Inclusion and Exclusion (PIE) with 3 variables



PIE occurs in *dual form* (the proofs are not given, but drawing a **Venn diagram** may help you understand what is going on):

$$(1) |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$
$$|A' \cap B' \cap C'| = |U| - |A| - |B| - |C| + |A \cup B| + |B \cup C| + |C \cup A| - |A \cup B \cup C|$$

(2) If A, B, C are events in the sample space U, then the probabilities can be calculated as:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$
$$P(A' \cap B' \cap C') = 1 - P(A) - P(B) - P(C) + P(A \cup B) + P(B \cup C) + P(C \cup A) - P(A \cup B \cup C)$$

Example 5 :

In a string of 180 cm long, every 3 cm are marked, every 4 cm are marked and every 5 cm are marked. If the string is cut according to all markings, how many pieces of strings are there after the cutting?

A = {sections of string with cutting according to every 3 cm marking only}

B = {sections of string with cutting according to every 4 cm marking only}

C = {sections of string with cutting according to every 5 cm marking only}

$$|A| = \frac{180}{3} = 60, \quad |B| = \frac{180}{4} = 45, \quad |C| = \frac{180}{5} = 36$$

$$|A \cap B| = \frac{180}{12} = 15, \quad |B \cap C| = \frac{180}{20} = 9, \quad |C \cap A| = \frac{180}{15} = 12$$

$$|A \cap B \cap C| = \frac{180}{60} = 3$$

$$\therefore |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$
$$= 60 + 45 + 36 - 15 - 9 - 12 + 3 = \underline{108}$$

\therefore After cutting, there are 108 pieces of strings.

Example 5 :

In a games day, Queen's College awarded 46 medals in football, 24 in basketball and 28 in table tennis. If these medals went to a total of 66 students, and only 4 students got medals in all the three sports, how many students received medals in exactly two of these sports?

Solution:

Let $|A|$ denote the number of students who was awarded football medals,
 $|B|$ denote the number of students who was awarded basketball medals,
 $|C|$ denote the number of students who was awarded table tennis medals.

Since $|A| = 46, |B| = 24, |C| = 28$

Also $|A \cup B \cup C| = 66$ and $|A \cap B \cap C| = 4$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

We have substituting these values we get,

$$66 = 46 + 24 + 28 - |A \cap B| - |B \cap C| - |C \cap A| + 4$$

$$|A \cap B| + |B \cap C| + |C \cap A| = 46 + 24 + 28 + 4 - 66 = 36$$

However, number of students awarded both football and basketball medals only

$$= |A \cap B| - |A \cap B \cap C|$$

number of students awarded both basketball and table tennis medals only

$$= |B \cap C| - |A \cap B \cap C|$$

number of students awarded both table tennis and football medals only

$$= |C \cap A| - |A \cap B \cap C|$$

Adding, we get:

Number of students received medals in exactly two of the three sports

$$= |A \cap B| + |B \cap C| + |C \cap A| - 3|A \cap B \cap C|$$

$$= 36 - 3(4) = \underline{\underline{24}}$$

Principle of Inclusion and Exclusion (PIE) with more than 3 variables

PIE can be extended to more variables. The general formula for **PIE** for n variables is

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{i=1}^n |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \dots + (-1)^{n-1} |A_1 \cap \dots \cap A_n|.$$

You may forget the formula if you find it difficult, but the idea of adding and subtracting is the same, as illustrated by the following example.

Example 6 : (Letter and envelope problem)

There are 5 different letters and 5 corresponding envelopes. If the letters are placed randomly in the envelopes, what is the probability that none of the letters are placed correctly in the envelopes and number of arrangements with all incorrect placements?

Solution

Let A be the event that letter A be correctly in its envelope, B be the event that letter B be correctly in its envelope, ..., E be the event that letter E be correctly in its envelope

$$P(A) = P(B) = P(C) = P(D) = P(E) = \frac{1}{5}$$

$$P(A \cap B) = \frac{1}{5} \times \frac{1}{4} = P(B \cap C) = \dots$$

$$\begin{aligned} P(A \cup B \cup C \cup D \cup E) &= P(A) + P(B) + P(C) + P(D) + P(E) \\ &\quad - P(A \cap B) - P(B \cap C) - \dots \\ &\quad + P(A \cap B \cap C) + P(B \cap C \cap D) + \dots \\ &\quad - P(A \cap B \cap C \cap D) - P(B \cap C \cap D \cap E) - \dots \\ &\quad + P(A \cap B \cap C \cap D \cap E) \\ &= 5 \times \frac{1}{5} - C_2^5 \times \frac{1}{5} \times \frac{1}{4} + C_3^5 \times \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} - C_4^5 \times \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} + \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times \frac{1}{1} \\ &= 1 - \frac{5 \times 4}{2 \times 1} \times \frac{1}{5 \times 4} + \frac{5 \times 4 \times 3}{3 \times 2 \times 1} \times \frac{1}{5 \times 4 \times 3} - \frac{5 \times 4 \times 3 \times 2}{4 \times 3 \times 2 \times 1} \times \frac{1}{5 \times 4 \times 3 \times 2} + \frac{1}{5 \times 4 \times 3 \times 2 \times 1} \\ &= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} \end{aligned}$$

Probability that none of the letters are placed correctly in the envelopes

$$\begin{aligned} &= 1 - P(A \cup B \cup C \cup D \cup E) \\ &= 1 - \left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} \right) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} = \frac{44}{120} = \underline{\underline{\frac{11}{30}}} \end{aligned}$$

Number of arrangements with all incorrect placements

$$= 5! \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \right) = \underline{\underline{44}}$$

Small Exercise

1. How many different numbers can be obtained from the product of two or more of the numbers :
3, 4, 5, 5, 6, 7, 7, 7?
2. How many 8-digit phone numbers contain at least one of each odd digit?